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Lift-and-project ranks of the set covering polytope of circulant matrices[☆]

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ABSTRACT

In this paper, we analyze the behavior of the N and N_+ operators (defined by Lovász and Schrijver) and the disjunctive operator due to Balas, Ceria and Cornuéjols, on the linear relaxation of the set covering polytope associated with circulant matrices C_n^k. We found that for the family of circulant matrices C_{sk+1}^k the disjunctive rank coincides with the N- and N_+-rank at the value $k-1$. This result provides bounds for lift-and-project ranks of most circulant matrices since C_{sk+1}^k appears as a minor of almost all circulant matrices. According to these operators, we define the strength of facets with respect to the linear relaxation of the set covering polytope and compare the results with a similar measure previously defined by Goemans. We identify facets of maximum strength although the complete description of the set covering polytope of circulant matrices is still unknown. Moreover, considering the matrices C_{sk}^k with $s \geq k+1$, we found a family of facets of the corresponding set covering polyhedron, having maximum strength according to the disjunctive and Goemans' measures.

1. Introduction

Let M be a 0, 1 matrix and consider the polyhedron

$$Q(M) = \{\mathbf{0} \leq x \leq \mathbf{1} : Mx \geq \mathbf{1}\}, \tag{1.1}$$

where $\mathbf{0}$ and $\mathbf{1}$ stand for the vector of all zeros and ones, respectively.

Every 0,1 solution of $Mx \geq \mathbf{1}$ is called a *cover* of M and the *covering number* of M denoted by $\tau(M)$ is the size of a minimum cover of M. If $Q(M)$ has only integer extreme points, the matrix M is called *ideal*. In this case, $Q(M)$ coincides with the *set covering polytope* $Q^*(M)$ defined as the convex hull of all covers of M. In general, $Q^*(M) \subset Q(M)$ and there are ways to quantify how far a matrix M is away from being ideal even though the description of $Q^*(M)$ is not known. In this context, we will consider *lift-and-project* operators.

Lift-and-project operators have been widely used in polyhedral combinatorics. Starting from a given polyhedron $\mathcal{K} \subset [0, 1]^n$, these methods attempt to give a description of the convex hull of integer solutions in $\mathcal{K}$, $\mathcal{K}^* = \mathrm{conv}(\mathcal{K} \cap \{0, 1\}^n)$, through a finite number of lift-and-project steps. In each step the current polyhedron (initially $\mathcal{K}$) is *lifted* to a higher dimensional space, where it is tightened, and then it is *projected* back. In [9], the authors introduce two such operators, N and N_+, by lifting the original polyhedron $\mathcal{K}$ to a higher dimensional space requiring about as many as the square of the original variables. Both operators obtain $\mathcal{K}^*$ in at most n steps, but one of them (N_+) combines linear restrictions with non-linear restrictions from the cone of semidefinite matrices.

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In [4] another lift-and-project operator, called *disjunctive* operator, requiring just about twice as many as the number of original variables in the lifting step, is presented. Although this operator generally obtains at each step a weaker relaxation than those of Lovász and Schrijver's, it also gets $\mathcal{K}^*$ in at most n steps.

Goemans introduced in [7] a notion for evaluating the *strength* of a linear relaxation for a combinatorial problem, relative to a weaker relaxation of that problem. He applied these results to compute the relative strength of classes of facet-defining inequalities for blocking-type polyhedra.

More precisely, given a relaxation $\mathcal{K}$ of $\mathcal{K}^*$ and a facet-defining inequality $ax \geq b$ for $\mathcal{K}^*$, Goemans defined the strength of the facet $ax \geq b$ with respect to $\mathcal{K}$ as $\frac{b}{\min\{ax : x \in \mathcal{K}\}}$. This measure is an indicator in comparing different classes of inequalities with respect to their potential effectiveness in a polyhedral cutting-plane algorithm.

Now, given a relaxation $\mathcal{K}$ of $\mathcal{K}^*$ and a facet-defining inequality $ax \geq b$ for $\mathcal{K}^*$, we can define the *lift-and-project strength* of a facet as its corresponding *rank*, i.e., the minimum number of steps the lift-and-project procedure in question needs, to obtain it as a valid constraint starting from $\mathcal{K}$. This idea is developed in Section 5.

In this paper we focus on polyhedra $Q(M)$ as in (1.1) when M is a consecutive-ones circulant matrix and study the number of steps these lift-and-project operators need in order to get the set covering polytope. A *consecutive-ones circulant* matrix is denoted by C_n^k and defined as a square matrix whose i-th row is the incidence vector of $\{i, i \oplus 1, \dots, i \oplus (k-1)\}$ for $i \in \{1, \dots, n\}$, where $2 \leq k \leq n-2$ and $\oplus$ denotes addition modulo n. Although we work with addition modulo n, throughout this paper we consider the set $\{1, \dots, n\}$ instead $\{0, \dots, n-1\}$. For the sake of simplicity C_n^k will be called circulant matrix.

The set covering polyhedron on these matrices has been studied in [1,3,2,6,10], among others.

For most circulant matrices C_n^k, the complete description of $Q^*(C_n^k)$ is not known. But it is not hard to see that the inequality,

$$\mathbf{1}x \geq \tau(C_n^k) = \left\lceil \frac{n}{k} \right\rceil, \tag{1.2}$$

is always valid for $Q^*(C_n^k)$ and following [10], it will be called the *rank constraint* associated with C_n^k. Also, in [10] it is proved that (1.2) defines a facet for $Q^*(C_n^k)$ if and only if n is not a multiple of k.

The purpose of this work is twofold. On the one hand, we study the lift-and-project operators on the set covering polytope associated with certain circulant matrices, proving that there is a particular family, C_{sk+1}^k, for which all the lift-and-project ranks coincide. The relevance of this family relies on the fact that almost all circulant matrices have a minor of this class. In Section 2 we present some known results on circulant matrices and minors. In Section 3 the formal definition of lift-and-project procedures is introduced, as well as some important results on their behavior over convex sets in $[0, 1]^n$. In Section 4 we study the minimum number of steps any of the lift-and-project procedures needs in order to get the set covering polytope associated with a circulant matrix. This is defined as the *lift-and-project rank* of the relaxation $Q(C_n^k)$. We also obtain an upper bound for the lift-and-project ranks of the set covering polytope of all circulant matrices.

On the other hand, in Section 5 we compare the strength of the various relaxations obtained through the lift-and-project operators and the measures utilized by Goemans, as suggested by Tunçel in [11]. In particular, we prove that the rank constraint associated with C_{sk+1}^k is a facet of maximum strength according to lift-and-project and Goemans' measure. Finally, using this result we identify a family of facets for $Q^*(C_{sk}^k)$ when $s \geq k+1$, having maximum strength according to the disjunctive operator and Goemans' measure.

2. Preliminaries

From now on, M is $m \times n$0, 1 matrix. If $i, j \in \{1, \dots, m\}$ and M^i, M^j are the i-th and j-th rows of M, respectively, we say that row i *dominates* row j if $M^i \leq M^j$.

We denote by M/j the *contraction* of column j, that is, column j is removed from M as well as the resulting dominated rows and hence, it corresponds to setting $x_j = 0$ in the constraints $Mx \geq \mathbf{1}$. The *deletion* of column j, denoted by $M \setminus j$ means that column j is removed from M as well as all the rows with a 1 in column j. This corresponds to setting $x_j = 1$ in the constraints $Mx \geq \mathbf{1}$. Given M and disjoint sets $V_1, V_2 \subset \{1, \dots, n\}$, contraction of all the columns indexed in V_1 and deletion of all the columns in V_2 can be performed sequentially and the resulting matrix does not depend on the order of indices or matrix operations. Then we say that $M/V_1 \setminus V_2$ is a *minor* of M and it is a *proper* minor if V_1 or V_2 are nonempty sets.

Remark 2.1. It is known (see [6]) that given a circulant matrix C_n^k every minor obtained by deletion is ideal; i.e. $Q(C_n^k) \cap \{x : x_j = 1\}$ is an integer polyhedron for every $j = 1, \dots, n$.

For each C_n^k, Cornuéjols and Novick introduced in [6] the directed graph $G(C_n^k)$ with vertices $V = \{1, \dots, n\}$ and such that (i, j) is an arc of $G(C_n^k)$ if $j \in \{i \oplus k, i \oplus (k+1)\}$.

Moreover, under this definition in [6, Lemma 4.5] it is shown that:

Lemma 2.2. *If $D \subset \{1, \dots, n\}$ induces a simple directed cycle in $G(C_n^k)$, then there exist nonnegative integer numbers n_1, n_2, n_3 with $n_1 \geq 1$ such that*

1. $nn_1 = kn_2 + (k+1)n_3$,
2. $\gcd(n_1, n_2, n_3) = 1$,
3. *if* $k - n_1 \leq 0$, C_n^k/D *is a zero matrix. If* $k - n_1 \geq 1$ *then* C_n^k/D *is isomorphic to* $C_{n-n_2-n_3}^{k-n_1}$.

In addition, in [1] it is obtained a characterization of contractions of circulant matrices that give back circulant matrices. Actually:

Theorem 2.3. *Let* $2 \leq k \leq n-1$, $D \subset \{1, \dots, n\}$ *with* $|D| = m$ *and* $0 \leq m \leq n-2$. *If* $0 \leq n_1 < \min\{k, m\}$ *then* C_n^k/D *is isomorphic to* $C_{n-m}^{k-n_1}$ *if and only if there exist* $d = \gcd(m, n_1)$ *disjoint simple directed cycles in* $G(C_n^k)$, *induced by* $D_1, \dots, D_d$ *each having length* m/d *such that* $D = \bigcup_r D_r$.

Remark 2.4. Let $s \geq 2$ and $k \geq 3$. If $D = \{1, 1+k, 1+2k, \dots, 1+(s-1)k\}$, using Lemma 2.2 we have that $n_1 = 1$, $n_2 = s-1$ and $n_3 = 1$ and it follows that $C_{s(k-1)+1}^{k-1}$ is a minor of C_{sk+1}^k. Analogously, if $0 \leq l < k-1$, $s \geq (l+1)k+1$ and

$$D = \{1, 1+k, \dots, 1+\mu k, 1+\mu k+(k+1), \dots, 1+\mu k+(\psi-1)(k+1)\},$$

for $\mu = s - k(l+1) - 1$ and $\psi = k(l+1) - l$, it holds that $n_1 = 1$, $n_2 = \mu$, $n_3 = \psi$ and $C_{s(k-1)+1}^{k-1}$ is a minor of C_{sk-l}^k.

3. Lift-and-project operators

In [4], the authors present a lift-and-project operator defined on polytopes $\mathcal{K}$ in $[0,1]^n$. For $j \in \{1, \dots, n\}$, the polytope $P_j(\mathcal{K})$ obtained after one lift-and-project iteration can be described as

$$P_j(\mathcal{K}) = \mathrm{conv}(\mathcal{K} \cap \{x \in \mathbb{R}_+^n : x_j \in \{0, 1\}\}).$$

Given a subset $F = \{i_1, \dots, i_k\}$ of $\{1, \dots, n\}$, in [4] it is proved that this operator can be applied iteratively over F, thus denoting the polytope $P_{i_1}(P_{i_2}(\dots P_{i_k}(\mathcal{K})))$ as $P_F(\mathcal{K})$. Clearly, $P_{\{1,\dots,n\}}(\mathcal{K}) = \mathcal{K}^*$ and the *disjunctive rank* of $\mathcal{K}$, $r_D(\mathcal{K})$, can be defined as the smallest cardinality of $F \subset \{1, \dots, n\}$ for which $P_F(\mathcal{K}) = \mathcal{K}^*$.

Now, we briefly overview the operator N and N_+ introduced in [9]. Here the authors work with convex cones $\tilde{\mathcal{K}} \subset \mathbb{R}^{n+1}$, homogenizing the inequalities by introducing a variable x_0. Thus a vector $x \in \mathbb{R}^{n+1}$ is of the form $(x_0, x_1, \dots, x_n)$ and we will work with vectors satisfying $0 \leq x_i \leq x_0$ for all $i = 1, \dots, n$.

Given a convex cone $\tilde{\mathcal{K}}$ whose points satisfy the inequalities above, $\mathcal{M}(\tilde{\mathcal{K}})$ denotes the cone of symmetric matrices $Y \in \mathbb{R}^{(n+1)\times(n+1)}$ such that $\mathrm{diag}(Y) = Ye_0$, $Ye_i \in \tilde{\mathcal{K}}$ and $Y(e_0 - e_i) \in \tilde{\mathcal{K}}$ for $i = 1, \dots, n$, where e_i is the $(i+1)$-th unit vector in $\mathbb{R}^{n+1}$.

The cone $N(\tilde{\mathcal{K}})$ is defined as $N(\tilde{\mathcal{K}}) = \{Ye_0 : Y \in \mathcal{M}(\tilde{\mathcal{K}})\}$.

By requiring the matrices in $\mathcal{M}(\tilde{\mathcal{K}})$ to be also positive semidefinite, we obtain the cones $\mathcal{M}_+(\tilde{\mathcal{K}})$ and $N_+(\tilde{\mathcal{K}}) = \{Ye_0 : Y \in \mathcal{M}_+(\tilde{\mathcal{K}})\}$.

For simplicity, when we say we are *applying* the N or N_+ operator to a convex set $\mathcal{K} \subset [0,1]^n$ we mean that we consider the cone generated by vectors of the form $\binom{1}{x}$, where $x \in \mathcal{K}$, apply the corresponding operator, then take the intersection of this cone with $y_0 = 1$ and project it back onto $\mathbb{R}^n$. $N(\mathcal{K})$ and $N_+(\mathcal{K})$, respectively, stand for these final subsets of $[0,1]^n$.

We notice that $N(\mathcal{K})$ is a polyhedron, whereas, in general $N_+(\mathcal{K})$ is not. Clearly $N_+(\mathcal{K}) \subset N(\mathcal{K})$.

If we set $N^0(\mathcal{K}) = N_+^0(\mathcal{K}) = \mathcal{K}$, $N^r(\mathcal{K}) = N(N^{r-1}(\mathcal{K}))$ and $N_+^r(\mathcal{K}) = N_+(N_+^{r-1}(\mathcal{K}))$ for $r \geq 1$, in [9] it is proved that $N^n(\mathcal{K}) = N_+^n(\mathcal{K}) = \mathcal{K}^*$.

This property allows the definition of $r(\mathcal{K})$, the *N-rank* of $\mathcal{K}$, as the smallest integer r for which $N^r(\mathcal{K}) = \mathcal{K}^*$. The *$N_+$-rank* (denoted by $r_+(\mathcal{K})$) is defined in a similar way.

In [4] it is proved that for any $\mathcal{K} \subset [0,1]^n$, the above defined operators generate relaxations of $\mathcal{K}^*$ satisfying

$$\mathcal{K}^* \subset N_+(\mathcal{K}) \subset N(\mathcal{K}) \subset P_j(\mathcal{K}) \subset \mathcal{K},$$

for every $j = 1, \dots, n$.

Therefore,

$$r_+(\mathcal{K}) \leq r(\mathcal{K}) \leq r_D(\mathcal{K}). \tag{3.1}$$

In order to simplify the notation, when there is no need to distinguish between the operators N and N_+ we simply write $N_\sharp$. Similarly, when we write $r_\sharp(\mathcal{K})$ we refer to *any* of the ranks of a polyhedron $\mathcal{K}$ in (3.1).

The following fact about the behavior of lift-and-project operators is well-known [8].

Lemma 3.2. *Let F be any face of* $[0,1]^n$ *and* $\mathcal{K} \subset [0,1]^n$ *be a convex set. Then, for every* $k \geq 0$,

$$N_\sharp^k(\mathcal{K} \cap F) = N_\sharp^k(\mathcal{K}) \cap F.$$

Remark 3.3. Consider $\mathcal{K} = Q(C_n^k)$ and D as in Theorem 2.3. Then, if $C_{n'}^{k'}$ is a minor of C_n^k isomorphic to C_n^k/D there is a natural one-to-one correspondence between $Q(C_{n'}^{k'})$ and $Q(C_n^k/D)$. Moreover, from Lemma 3.2, this correspondence is preserved after the successive applications of the lift-and-project operators. Therefore, $r_\sharp(Q(C_{n'}^{k'})) \leq r_\sharp(Q(C_n^k))$.

In the following section we will focus our attention on matrices of the form C^k_{sk+1}, which, according to Remark 2.4, appear as minors of most circulant matrices.

4. Lift-and-project operators on circulant matrices

A general circulant matrix is defined through the *shift* operator T. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is such that $T(v_1, \ldots, v_n) = (v_n, v_1, \ldots, v_{n-1})$, then $\mathrm{circ}(u)$ is the $n \times n$ matrix whose first row is $T^0(u) = u$ and whose j-th row is given by $T^{j-1}(u)$, for every $j \geq 2$. It is easy to see that if $1 \leq k \leq n-2$ then $C^k_n = \mathrm{circ}(v^k)$ where $v^k \in \{0, 1\}^n$ and $v^k_i = 1$ if and only if $i \in \{1, \ldots, k\}$.

The following result provides an upper bound for the disjunctive rank of every circulant matrix.

Theorem 4.1. *If* $1 \leq k \leq n-2$ *then* $r_D(Q(C^k_n)) \leq k-1$.

Proof. If $k = 1$, then $Q(C^k_n)$ is an integer polyhedron. Let $F = \{1, \ldots, k-1\}$ with $k \geq 2$ and x be an extreme point of $P_F(Q(C^k_n))$. If $x_i = 1$ for some $i \in F$ then Remark 2.1 implies that x is an integer point. Otherwise, $x \in Q(C^k_n) \cap \{x : x_j = 0, j \in F\}$ and then $x_k = 1$. Again, Remark 2.1 shows that x is an integer point. Therefore, $P_F(Q(C^k_n)) = Q^*(C^k_n)$. □

Concerning the N_+-operator, it is proved in [5] that $r_+(Q(C^k_{2k+1})) \geq k-1$ for $k \geq 2$. After inequality (3.1) and Theorem 4.1 we have:

Corollary 4.2. *For* $k \geq 2$, $r_\sharp(Q(C^k_{2k+1})) = k-1$.

Next we prove an extension of the result in [5] for any $s \geq 2$.

Theorem 4.3. *If* $s \geq 2$ *and* $k \geq 2$ *then* $r_+(Q(C^k_{sk+1})) \geq k-1$.

Proof. Let $s \geq 2$, $k \geq 2$, and consider $x^k = \alpha_k \mathbf{1} \in \mathbb{R}^{sk+1}$, where

$$\alpha_k = \begin{cases} \dfrac{1}{2} & \text{if } k = 2, \\[2ex] \dfrac{\alpha_{k-1}(1 + s(k-1))}{\alpha_{k-1}(1 + s(k-1)) + s(k-1)} & \text{if } k \geq 3. \end{cases}$$

By induction on k, it is not hard to check that $x^k \in Q(C^k_{sk+1})$ and that it violates the rank inequality (1.2), i.e.,

$$\mathbf{1}x^k < s + 1 = \left\lceil \frac{sk+1}{k} \right\rceil.$$

Then, it follows that $x^k \in Q(C^k_{sk+1}) \setminus Q^*(C^k_{sk+1})$.

Let us show that $x^k \in N^{k-2}_+(Q(C^k_{sk+1}))$, thus proving the desired result. For $k = 2$ this is clear since $N^0_+(Q(C^2_{2s+1})) = Q(C^2_{2s+1})$.

For $k \geq 3$, assume that $x^{k-1} \in N^{k-3}_+(Q(C^{k-1}_{s(k-1)+1}))$ and consider the $(sk+2) \times (sk+2)$ matrix Y^k of the form

$$Y^k = \begin{pmatrix} 1 & (x^k)^T \\ x^k & \tilde{Y}^k \end{pmatrix}$$

where $\tilde{Y}^k \in \mathbb{R}^{(sk+1)\times(sk+1)}$ is defined as

$$\tilde{Y}^k_{ij} = \begin{cases} \alpha_k & \text{if } i = j, \\[1ex] \alpha_k \dfrac{(k-1)(s-l)+1}{s(k-1)+1} & \text{if } j = i \oplus lk, i \oplus (-lk) \text{ and } l = 1, \ldots, s-1, \\[2ex] \alpha_k \dfrac{1}{s(k-1)+1} & \text{otherwise.} \end{cases}$$

It is easy to see that Y^k is a symmetric matrix and that $\mathrm{diag}(\tilde{Y}^k) = x^k$.

Claim 1. *If* $i \in \{1, \ldots, sk+1\}$ *then* $\frac{1}{\alpha_k}\tilde{Y}^k e_i \in N^{k-3}_+(Q(C^k_{sk+1}))$.

Proof. Let us observe that a point x belongs to $Q(C^k_{sk+1})$ if and only if $\sum_{j=0}^{k-1} x_{r\oplus j} \geq 1$ for every $r = 1, \ldots, sk+1$. If we call $\beta_l = \frac{(k-1)(s-l)+1}{s(k-1)+1}$ for $l = 1 \ldots, s$, then $\frac{1}{\alpha_k}\tilde{Y}^k e_i$ satisfies the previous restrictions since:

(i) if $r = i + 1$ or $r = i \oplus (s-1)k + 1$ then

$$\frac{1}{\alpha_k}\sum_{j=0}^{k-1}(\tilde{Y}^k e_i)_{r\oplus j} = \beta_1 + (k-1)\beta_s = 1$$

(ii) if $r = i \oplus ((s-1)k+2), \ldots, i \oplus (sk+1)$ then

$$\frac{1}{\alpha_k}\sum_{j=0}^{k-1}(\tilde{Y}^k e_i)_{r\oplus j} = 1 + (k-1)\beta_s > 1$$

(iii) if $r = i \oplus ((l-1)k+2), \ldots, i \oplus (lk)$ with $l = 1, \ldots, s-1$ then

$$\frac{1}{\alpha_k}\sum_{j=0}^{k-1}(\tilde{Y}^k e_i)_{r\oplus j} = (k-2)\beta_s + \beta_l + \beta_{s-l} = 1 + (k-1)\beta_s > 1$$

(iv) if $r = i \oplus (lk+1)$ with $l = 1, \ldots, s-2$ then

$$\frac{1}{\alpha_k}\sum_{j=0}^{k-1}(\tilde{Y}^k e_i)_{r\oplus j} = (k-2)\beta_s + \beta_{s-l} + \beta_{l+1} = 1$$

Then, $\frac{1}{\alpha_k}\tilde{Y}^k e_i \in Q(C^k_{sk+1}) \cap \{x: x_i = 1\}$. According to Remark 2.1, $Q(C^k_{sk+1}) \cap \{x: x_i = 1\}$ is an integer polyhedron and then

$$\frac{1}{\alpha_k}\tilde{Y}^k e_i \in Q^*(C^k_{sk+1}) \subset N_+^{k-3}(Q(C^k_{sk+1})).$$

Thus, the claim holds. □

Claim 2. *If* $i \in \{1, \ldots, sk+1\}$ *then* $\frac{1}{1-\alpha_k}(x^k - \tilde{Y}^k e_i) \in N_+^{k-3}(Q(C^k_{sk+1}))$.

Proof. Let $U_l = \{i, i\oplus k, \ldots, i\oplus(l-1)k, i\oplus(lk+1), \ldots, i\oplus((s-1)k+1)\}$ and $w^l \in \mathbb{R}^{sk+1}$ be defined as follows

$$w^l_j = \begin{cases}\alpha_{k-1} & \text{if } j \notin U_l,\\ 0 & \text{if } j \in U_l,\end{cases}$$

for $l = 1, \ldots, s$.

Recall that after Remark 2.4, $C^{k-1}_{s(k-1)+1}$ is a minor of C^k_{sk+1} obtained after the contraction of $D = U_l$ with parameters $n_1 = 1, n_2 = s-1$ and $n_3 = 1$. In other words, the zero entries of w^l are exactly the columns deleted to get the minor. Using Remark 3.3, with $\mathcal{K} = Q(C^k_{sk+1})$ and $D = U_l$ and induction hypothesis, i.e., $x^{k-1} = \alpha_{k-1}\mathbf{1} \in N_+^{k-3}(Q(C^{k-1}_{s(k-1)+1}))$, we have $w^l \in N_+^{k-3}(Q(C^k_{sk+1}))$.

Moreover, the entries of $w = \frac{1}{s}\sum_{l=1}^{s} w^l$ are

$$w_j = \begin{cases}0 & \text{if } j = i,\\ \dfrac{l}{s}\alpha_{k-1} & \text{if } j = i\oplus lk, i\oplus(-lk) \text{ and } l = 1, \ldots, s-1,\\ \alpha_{k-1} & \text{otherwise.}\end{cases}$$

Since $\alpha_{k-1} = \frac{\alpha_k}{1-\alpha_k}\frac{s}{l}(1-\beta_l)$ for all $l = 1, \ldots, s$, we get $w = \frac{1}{1-\alpha_k}(x^k - \tilde{Y}^k e_i)$.

Therefore, by convexity the claim is proved. □

After Claims 1 and 2 and the definition of the N-operator, we have that $x^k \in N(N_+^{k-3}(Q(C^k_{sk+1})))$.

It remains to prove that Y^k is a PSD matrix. It is enough to show that the Schur complement $\tilde{Y}^k - x^k(x^k)^T$ is PSD (for further details see [11]).

Now, $\tilde{Y}^k - x^k(x^k)^T = \alpha_k[\frac{1}{\alpha_k}\tilde{Y}^k - \alpha_k E]$ where E is the $n \times n$ matrix with all entries at value one. Moreover, $\frac{1}{\alpha_k}\tilde{Y}^k - \alpha_k E = \mathrm{circ}(z)$ if $z = \frac{1}{\alpha_k}\tilde{Y}^k e_1 - \alpha_k\mathbf{1}$.

It is known that, if $\epsilon_j = \exp(\frac{2\pi i}{sk+1}j)$ for $j = 0, \ldots, sk$, then the eigenvalues of $\mathrm{circ}(z)$ are $\lambda_j = \sum_{m=0}^{sk} z_{m+1}\epsilon_j^m$. Recall that ϵ_j is an $(sk+1)$-th root of the unity. Then, $\epsilon_j^{sk+1} = 1$ and, if $j \neq 0$, $\sum_{m=0}^{sk}\epsilon_j^m = 0$.

Let us call $\gamma = 1-\alpha_k(s(k-1)+1)$ and $\delta_r = (s-r)(k-1)$ for $r = 1, \ldots, s-1$, then we can write $(s(k-1)+1)z = u + \gamma\mathbf{1}$ where

$$u_j = \begin{cases}s(k-1) & \text{if } j = 1,\\ \delta_r & \text{if } j = 1\oplus rk, 1\oplus(-rk) \text{ and } r = 1, \ldots, s-1,\\ 0 & \text{otherwise.}\end{cases}$$

For every $j = 1, \dots, sk$, we obtain that

$$\begin{aligned}(s(k-1)+1)\lambda_j &= \sum_{m=0}^{sk} u_{m+1}\epsilon_j^m + \gamma \sum_{m=0}^{sk} \epsilon_j^m \\ &= \sum_{m=0}^{sk} u_{m+1}\epsilon_j^m \\ &= s(k-1) + \sum_{r=1}^{s-1} \delta_r(\epsilon_j^{rk} + \epsilon_j^{-rk}) \\ &= (k-1)\left(s + \sum_{r=1}^{s-1}(s-r)2\cos\left(\frac{2\pi jkr}{sk+1}\right)\right)\end{aligned}$$

where in the last equality we used the fact that $\exp(ix) = \cos(x) + i\sin(x)$ to get $\epsilon_j^m + \epsilon_j^{-m} = 2\cos\left(\frac{2\pi mj}{sk+1}\right)$ for $j, m \in \{0, \dots, sk\}$.

For $j = 1, \dots, sk$ it can be proved by induction that

$$(s(k-1)+1)\lambda_j = \frac{(k-1)\sin^2\left(\frac{\pi jsk}{sk+1}\right)}{\sin^2\left(\frac{\pi jk}{sk+1}\right)}.$$

In addition, $\lambda_0 = \sum_{m=0}^{sk} z_{m+1} = 1 - \alpha_k(1+sk) + s$ and it is positive since $(sk+1)\alpha_k < s+1$. Therefore, $\lambda_j > 0$ for every $j = 0, \dots, sk$ and then Y^k is a PSD matrix. □

As an immediate consequence of Theorems 4.1 and 4.3, we have:

Corollary 4.4. *If $s \geq 2$ and $k \geq 2$ then $r_\sharp(Q(C^k_{sk+1})) = k-1$.*

Using Remarks 2.4 and 3.3, Theorem 4.1 and Corollary 4.4 we also have:

Corollary 4.5. *If $k \geq 3$, $0 \leq l < k-1$ and $s \geq (l+1)k+1$ then $k-2 \leq r_\sharp(Q(C^k_{sk-l})) \leq k-1$.*

The following result relates the disjunctive rank of the set covering polytope of a circulant matrix and the ranks of its minors.

Lemma 4.6. *Let $C^{k'}_{n'}$ be a nonideal proper minor of C^k_n. If $r_D(Q(C^{k'}_{n'})) \geq p$ then $r_D(Q(C^k_n)) \geq p+1$.*

Proof. Let $F = \{i_1, \dots, i_p\} \subset \{1, \dots, n\}$. We will show that $P_F(Q(C^k_n))$ is not an integral polyhedron, which leads to the desired result. By Remark 2.1, a nonideal proper minor can only be obtained by contraction. Therefore, there is a set $D \subset \{1, \dots, n\}$ such that C^k_n/D is isomorphic to $C^{k'}_{n'}$. Moreover, w.l.o.g. we can consider that $F \cap D \neq \emptyset$. Then $F \setminus D$ indexes a subset of columns of matrix C^k_n/D with cardinality at most $p-1$.

Under the assumption that $r_D(C^{k'}_{n'}) \geq p$, there must be a fractional extreme point $\bar{x}$ in $P_{F\setminus D}Q(C^k_n/D)$ having all its components in $F \setminus D$ at value zero. Now, let $x \in \mathbb{R}^n$ be such that

$$x_i = \begin{cases} 0 & \text{if } i \in F \cup D, \\ \bar{x}_i & \text{otherwise.} \end{cases}$$

Therefore, x is a fractional extreme point of $P_F(Q(C^k_n))$ and this shows that $r_D(Q(C^k_n)) \geq p+1$. □

Moreover, using this result we are able to prove that:

Corollary 4.7. *If $k \geq 3$, $0 \leq l < k-1$ and $s \geq (l+1)k+1$ then $r_D(Q(C^k_{sk-l})) = k-1$.*

Proof. From Remark 2.4 we have that $C^{k-1}_{s(k-1)+1}$ is a minor of C^k_{sk-l} and by Corollary 4.4, $r_D(Q(C^{k-1}_{s(k-1)+1})) = k-2$. Then, from Lemma 4.6 and Theorem 4.1, the result follows. □

5. Strength of facets

According to the results in [2] for most nonideal circulant matrices, the rank constraint is not the only one needed for a description of the set covering polyhedron. Moreover, in [2] a family of non-rank facet inequalities for $Q^*(C^k_n)$ is presented. Actually:

Theorem 5.1. *Let C_n^k be a nonideal circulant matrix and let $D \subset \{1, \ldots, n\}$ induce a simple directed cycle in $G(C_n^k)$ such that C_n^k/D is isomorphic to $C_{n'}^{k'}$ and $n' \equiv 1 (\mathrm{mod}\ k')$, $k' \geq 2$. Let V_0, V_T be a partition of D such that $i \in V_T$ if and only if $(i-k-1, i)$ is an arc of the cycle. Then, the inequality*

$$\sum_{i \notin D} x_i + \sum_{i \in V_0} x_i + 2 \sum_{i \in V_T} x_i \geq \left\lceil \frac{n'}{k'} \right\rceil, \tag{5.2}$$

defines a facet of $Q^*(C_n^k)$ if and only if $\left\lceil \frac{n'}{k'} \right\rceil > \left\lceil \frac{n}{k} \right\rceil$.

In [3] it is studied Goemans' strength of inequalities (5.2) when they define facets of $Q^*(C_n^k)$. In particular:

Lemma 5.3. *Let $k \geq 3$ and $s \geq 1$.*

(1) *The facet-defining inequality of maximum strength of $Q^*(C_{sk+1}^k)$ with respect to $Q(C_{sk+1}^k)$, is the rank constraint.*
(2) *If $s \geq k+1$, the facet-defining inequalities of maximum strength of $Q^*(C_{sk}^k)$ with respect to $Q(C_{sk}^k)$, are the inequalities given by (5.2) where D is any subset of $\{1, \ldots, sk\}$ such that C_{sk}^k/D is isomorphic to $C_{(k-1)s+1}^{k-1}$.*

In this section we analyze the strength of the facets considered in the previous lemma according to the lift-and-project procedures.

Firstly, if $\mathcal{K}$ is a linear relaxation of $\mathcal{K}^*$ and L is any of the lift-and-project operators introduced in Section 3, we say that the *L-rank* of a facet constraint $ax \geq b$ of $\mathcal{K}^*$ according to L, is the minimum number of steps r needed to obtain $ax \geq b$ as a valid inequality for $L^r(\mathcal{K})$.

According to this definition, the *L-strength* of a facet of $\mathcal{K}^*$ is its corresponding L-rank. Trivially, an inequality of maximum strength is one having the lift-and-project rank of $\mathcal{K}$.

Observe that in the proof of Theorem 4.3 we show that $r_+(Q(C_{sk+1}^k)) = k-1$ by presenting a point $x^k \in N_+^{k-2}(Q(C_{sk+1}^k))$ that violates the rank constraint. Then, it is a facet of maximum strength according to the N_+ operator. This means that it is also a facet of maximum strength for the disjunctive and N operator. Therefore,

Theorem 5.4. *If $s \geq 2$ and $k \geq 2$ then the rank constraint (1.2) is a facet of maximum L-strength of $Q^*(C_{sk+1}^k)$ where L stands for the disjunctive, N or N_+ operator.*

Now, consider the family of circulant matrices C_{sk}^k for $s \geq k+1$.

Theorem 5.5. *If $k \geq 3$ and $s \geq k+1$, the inequalities given by (5.2), where D is any subset of $\{1, \ldots, sk\}$ such that C_{sk}^k/D is isomorphic to $C_{(k-1)s+1}^{k-1}$, are facet-defining inequalities of maximum disjunctive strength of $Q^*(C_{sk}^k)$.*

Proof. If D is such that C_{sk}^k/D is isomorphic to $C_{(k-1)s+1}^{k-1}$, consider $W = \{i \in D : i-k-1 \in D\}$. Then, the inequality in (5.2) can be rewritten as

$$\sum_{i \notin W} x_i + 2 \sum_{i \in W} x_i \geq s+1. \tag{5.6}$$

From Corollary 4.7, $r_D(C_{sk}^k) = k-1$. Then, it is enough to show that for every set F with $|F| \leq k-2$ there is a point in $P_F(Q(C_{sk}^k))$ that violates the inequality (5.6).

As we have already done in the proof of Lemma 4.6, consider $F = \{i_1, \ldots, i_{k-2}\} \subset \{1, \ldots, sk\}$ and the $D \subset \{1, \ldots, sk\}$ such that C_{sk}^k/D is isomorphic to $C_{s(k-1)+1}^{k-1}$ with $F \cap D \neq \emptyset$. Then $F \setminus D$ has at most $k-3$ elements.

By Corollary 4.4, $r_D(Q(C_{s(k-1)+1}^{k-1})) = k-2$ and from Theorem 5.4 the rank inequality is a facet of maximum disjunctive strength. Therefore, there is a fractional extreme point $\bar{x}$ in $Q(C_{sk}^k/D)$ having all its components in $F \setminus D$ at value zero that violates the rank inequality associated with $C_{s(k-1)+1}^{k-1}$, i.e. $\sum_{i \notin F \cup D} \bar{x}_i < s+1$.

Let $x \in \mathbb{R}^n$ be such that

$$x_i = \begin{cases} 0 & \text{if } i \in F \cup D, \\ \bar{x}_i & \text{otherwise.} \end{cases}$$

Then, $x \in P_F(Q(C_{sk}^k))$ and violates inequality (5.6) since $W \subseteq D$. $\square$

6. Conclusions and open problems

In this paper we analyzed the behavior of the disjunctive, N and N_+ operators over a wide family of circulant matrices. We found in Theorem 4.1 an upper bound for all the lift-and-project ranks of the set covering polyhedron on circulant matrices. Also, we could compute all the lift-and-project ranks over the family C_{sk+1}^k providing lower bounds for the lift-and-project ranks for the linear relaxation of the set covering polytope of most circulant matrices.

Although the set covering polyhedra for matrices C^k_{sk+1} and C^k_{sk} with $s \geq k+1$ are not known, we could identify facets of maximum strength of these polyhedra when we consider Goemans' and lift-and-project measures defined in Section 5. Moreover, we have proved they are facets of maximum strength according to any of them.

On the other hand, it is known that the set packing polyhedron on circulant matrices can be stated in terms of the stable set polytope on web graphs. Our future work consists in studying the behavior of lift-and-project procedures on the clique relaxation of these graphs and compare the strength of facets according to Goemans' and lift-and-project measures. This would complete the line of research suggested by Tunçel in [11].

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